LATERAL EDGE EFFECT IN ELECTRODIFFUSION MEASUREMENTS OF WALL SHEAR STRESS

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Received August 25th, 1987
Accepted December 30th, 1987

The influence of three-dimensional diffusion at lateral edges of an electrodiffusion sensor of friction is analysed by the method of singular perturbations for $Pe_L \gg 1$. It is shown that lateral diffusion at $Pe_L \gg 1$ has a more pronounced effect than longitudinal diffusion at the back edge of the sensor, which has hitherto been exclusively considered.

Considerable attention has recently been paid to electrodiffusion methods of measurement of the shear stress between a streaming liquid and a wall[1-3]. These are based on the theory of convective diffusion in the diffusion layer approximation, neglecting the effect of longitudinal and lateral diffusion. Estimates of the influence of longitudinal diffusion at the front and back edges of an infinitely wide band electrode are given in the literature[4-11]. The influence of lateral diffusion to electrodes of finite width was, so far as known to us, not treated in the literature.

THEORETICAL

Formulation of the Problem

The transport configuration considered is shown in Fig. 1. A Newtonian liquid flows along a plate, $z = 0$, under conditions at which the velocity profile in the diffusion layer, $0 < z < \delta(x)$, is linear:

$$v_x = \gamma z\,, \quad v_y = 0\,, \quad v_z = 0\,. \tag{1}$$

A rectangular electrode for electrodiffusion measurements of length L, $0 < x < L$, and width h is inserted in the plate. The problem about steady convective diffusion of a depolarizer from the bulk of the flowing liquid $(c = c_0)$ to the electrode surface $(c = 0)$ is described by the following elliptic equation of convective diffusion for the concentration field $c = c(x, y, z)$

$$\gamma z \partial_x c = D(\partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz})\, c \tag{2}$$

with the boundary conditions in the bulk of the liquid

$$c \to c_0 \quad \text{for} \quad \begin{cases} z/\delta \to \infty \\ (x^2 + y^2)/(L^2 + h^2) \to \infty \end{cases} \tag{3}$$

at the electrode surface, $\mathscr{W}$

$$c = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad (x, y) \in \mathscr{W} \tag{4}$$

and at the insulating plate surface, $\mathscr{N}$

$$\partial_z c = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad (x, y) \in \mathscr{N} . \tag{5}$$

Classical Lévêque's solution to this problem in the diffusion layer approximation, i.e. neglecting the influence of longitudinal, $D\partial^2_{xx}c$, and lateral, $D\partial^2_{yy}c$, diffusion, can be written in the form

$$c \approx c_L \equiv \frac{c_0}{\Gamma(4/3)} \int_0^{zx^{-1/3}\lambda^{-2/3}} \exp(-s^3)\, ds , \tag{6}$$

where

$$\lambda = (9D/\gamma)^{1/2} . \tag{7}$$

Based on this solution, the local current densities can be expressed by the equation (F denotes Faraday's constant)

$$J_L = Fc_0 D/\delta \tag{8}$$

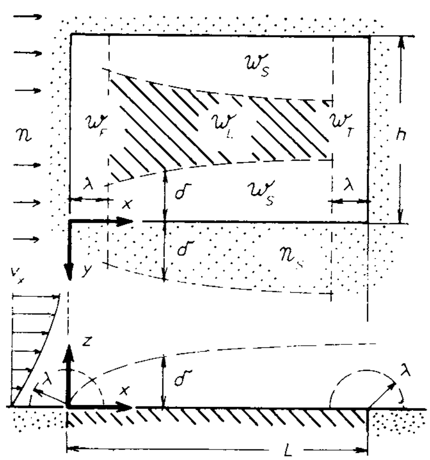


FIG. 1

Electrodiffusion friction sensor of rectangular cross section. The characteristic three-dimensional transport regions are indicated: $\mathscr{N}_S$ denotes outer region of the lateral edge; $\mathscr{N}$ insulation surface as a whole; $\mathscr{W}_L$ Lévêque diffusion layer region; $\mathscr{W}_F$ inner region of the front edge; $\mathscr{W}_T$ inner region of the rear edge; $\mathscr{W}_S$ inner region of the lateral edge; $\mathscr{W} = \mathscr{W}_L + \mathscr{W}_S + \mathscr{W}_F + \mathscr{W}_T$ is the electrode surface; x, y, z are Cartesian coordinates; λ constant diffusion length, Eq. (7); $\delta = \delta(x)$ diffusion thickness, Eq. (9); h, L electrode width and length; v_x longitudinal velocity in which the mass flux is expressed by introducing the Nernst diffusion thickness in the Lévêque approximation

$$\delta(x) \equiv \frac{c_0}{\partial_z c_L|_{z=0}} \approx \Gamma(4/3)\, x^{1/3} \lambda^{2/3} \doteq 1{\cdot}857(xD/\gamma)^{1/3}\,. \qquad (9)$$

The domain on which this solution can be considered as a good local approximation of the concentration field is denoted in Fig. 1 as $\mathscr{W}_L$ and by shadowing. The width of the frontal, $\mathscr{W}_F$, and rear, $\mathscr{W}_T$, transition regions is characterized[6,11] by the length parameter, λ. The object of the present work is the determination of the concentration field and current densities in the lateral regions $\mathscr{W}_S$ and $\mathscr{N}_S$ where neither of the three components of the diffusion flux can be neglected and therefore the three-dimensional elliptic problem must be solved.

By transformation of the concentration field

$$C(X, Y, Z) = \Gamma(4/3)\, c(x, y, z)/c_0\,,$$
$$X = x\lambda^{-1}\,, \quad Y = yx^{-1/3}\lambda^{-2/3}\,, \quad Z = zx^{-1/3}\lambda^{-2/3} \qquad (10)$$

and neglecting the terms of the order of $O(X^{-4/3})$ for $X \to \infty$ it is possible to simplify Equation (2) to the following two-dimensional form

$$\partial^2_{YY}C + \partial^2_{ZZ}C + 3Z^2\partial_Z C + 3ZY\partial_Y C = 0 \qquad (11)$$

which for $Y\partial_Y C \to 0$, $\partial^2_{YY}C \to 0$ takes the form of the Lévêque diffusion layer approximation. An ad hoc condition is introduced

$$C \to \int_0^Z \exp(-s^3)\, \mathrm{d}s \quad \text{for} \quad Y \to -\infty\,, \qquad (12)$$

expressing the starting assumption of the perturbation analysis about continuous transition of the domain $\mathscr{W}_S$ into the domain $\mathscr{W}_L$ for $-y/\delta \to \infty$. Other conditions remain unchanged:

$$C \to \Gamma(4/3) \quad \text{for} \quad Z \to \infty \quad \text{or} \quad Y \to \infty\,, \qquad (13)$$

$$C = 0 \quad \text{for} \quad Z = 0 \quad \text{and} \quad Y < 0\,, \qquad (14)$$

$$\partial_Z C = 0 \quad \text{for} \quad Z = 0 \quad \text{and} \quad Y > 0\,. \qquad (15)$$

Conformal Regularization; Discrete Representation

The formulation of the problem by Equations (11)–(15) is not suitable for numerical solution by the finite-difference method, since the solution has a singularity at the point $Y = 0$ (compare[6,9,11]), characterized by the relation $\partial_Z C|_{Z=0} \sim (-Y)^{-1/2}$ for $Y \to 0_-$. This can be removed by conformal mapping of the coordinates, $(Y, Z) \to \to (\eta, \zeta)$, which transforms the electrode surface ($Z = 0$, $Y < 0$) and the insulating surface ($Z = 0$, $Y > 0$) to two planes perpendicular to each other[6,9,11] (Fig. 2):

$$C(Z, Y) = \Theta(\zeta, \eta)$$

$$\eta^2 - \zeta^2 = 2Y, \quad \eta\zeta = Z\,. \tag{16}$$

From Equations (*11*)–(*15*), the boundary value problem follows for the function $\Theta(\zeta, \eta)$, $\zeta \geqq 0$, $\eta \geqq 0$:

$$\partial^2_{\zeta\zeta}\Theta + \partial^2_{\eta\eta}\Theta + \tfrac{3}{2}\zeta\eta(\zeta^2 + \eta^2)(\zeta\partial_\zeta\Theta + \eta\partial_\eta\Theta) = 0\,, \tag{17}$$

$$\Theta \approx \int_0^{\zeta\eta} \exp(-s^3)\,\mathrm{d}s \quad \text{for} \quad \zeta > \zeta_W\,, \tag{18}$$

$$\Theta = \Gamma(4/3) \quad \text{for} \quad \eta > \eta_N\,, \tag{19}$$

$$\Theta = 0 \quad \text{for} \quad \eta = 0\,, \tag{20}$$

$$\partial_\zeta\Theta = 0 \quad \text{for} \quad \zeta = 0\,, \tag{21}$$

where ζ_W and η_N are sufficiently large numbers. At the electrode surface ($Y < 0$,

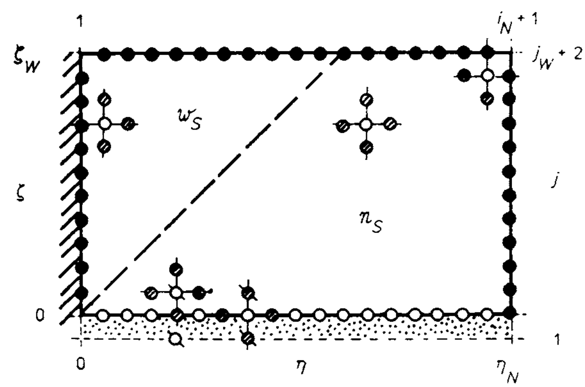


FIG. 2

Conformal mapping of the three-dimensional lateral region $\mathscr{W}_S + \mathscr{N}_S$ to the rectangle $(0; \zeta_W) \times (0; \eta_N)$ and the grid for the numerical solution. Electrode surface is denoted as $\eta = 0$; insulating surface as $\zeta = 0$; $\eta = \eta_N$ denotes boundary with the lateral concentration tail; $\zeta = \zeta_W$ boundary with the inner diffusion layer; dashed diagonal line $\zeta = \eta$ is the plane $y = 0$ over the lateral edge; solid circles denote the given boundary values; shaded circles denote values from the preceding iteration; open circles denote values calculated from Eq. (*25*); crossed points are pairs of identical values from Eq. (*37*)

$Z = 0$, i.e. $\eta = 0$, $\zeta > 0$) we have

$$\partial_Z C|_{Z=0} = \zeta^{-1} \partial_\eta \Theta|_{\eta=0} \,. \tag{22}$$

Hence it follows the expression for Ψ_S

$$\Psi_S \equiv \int_{-\infty}^{0} (\partial_Z C|_{Z=0} - 1) \, \mathrm{d}Y \approx \int_0^{\zeta_W} (\partial_\zeta \Theta|_{\zeta=0} - \zeta) \, \mathrm{d}\zeta \,, \tag{23}$$

which is of basic importance for the calculation of the diffusion current increment due to lateral diffusion. The values of ζ_W and η_N necessary for the attainment of the required accuracy must be determined by trial and error.

The above boundary value problem on the rectangle $\langle 0; \zeta_W \rangle \times \langle 0; \eta_N \rangle$ was solved by the finite-difference method. The step of the grid was constant in both directions:

$$\Theta_{i,j} = \Theta(\zeta_j, \eta_i) \,,$$
$$\eta_i = \alpha(i-1) \,, \quad i = 1, \ldots, i_N + 1 \,,$$
$$\zeta_j = \alpha(j-2) \,, \quad j = 1, \ldots, j_W + 2 \,. \tag{24}$$

Here, i_N and j_W give the number of steps inside the definition region. The differential equation (*17*) was discretized by using the standard representation of the partial derivatives by central differences[12]. The new estimate, $\Theta_{i,j}^{*k}$, based on the given values of $\Theta_{i,j}^k$ close to a given point is according to the four-point difference scheme given as

$$\Theta_{i,j}^{*k} = (\Theta_{i,j+1}^k + \Theta_{i,j-1}^k + \Theta_{i+1,j}^k + \Theta_{i-1,j}^k)/4 +$$
$$+ w_{i,j}[\zeta_j(\Theta_{i,j+1}^k - \Theta_{i,j-1}^k) + \eta_i(\Theta_{i+1,j}^k - \Theta_{i-1,j}^k)] \,, \tag{25}$$

where

$$w_{i,j} = (3\alpha/16) \, \zeta_j \eta_i (\zeta_j^2 + \eta_i^2) \,. \tag{26}$$

To express the condition $\partial_\zeta \Theta = 0$ on the insulating surface, $\zeta = 0, j = 2$, we introduce an auxiliary line, $\zeta_1 = -\alpha$, outside the definition region. The relation

$$\Theta_{3,i} = \Theta_{1,i} \tag{27}$$

then expresses the condition of mirror symmetry with respect to the plane $\zeta = 0$. The gradient at the electrode surface is calculated according to the asymmetrical three-point formula

$$(\partial_\eta \Theta|_{\eta=0})|_{\zeta=\zeta_j} = (4\Theta_{2,j} - \Theta_{3,j})/2\alpha \tag{28}$$

and Simpson's formula is used to calculate the integral (*23*).

The input estimate of the $\Theta_{i,j}$ field was made according to the formulae

$$\Theta(\zeta, \eta) \approx \begin{cases} \int_0^{\zeta\eta} \exp(-s^3)\, ds, & \zeta > \eta \quad (29a) \\ \int_0^{\zeta^2+\eta^2} \exp(-s^3)\, ds, & \zeta \leqq \eta \quad (29b) \end{cases}$$

which also define the asymptotical boundary conditions (*18*) and (*19*) and satisfy the boundary conditions (*20*) and (*21*).

Preliminary numerical experiments showed that to attain the required accuracy it is necessary to choose $i_N > 90$, $j_W > 60$. The iteration solution of the resulting system of more than 5 000 linear equations was carried out by the relaxation method using the formula

$$\Theta_{i,j}^{k+1} = \sigma_{ij}^k \Theta_{i,j}^{*k} + (1 - \sigma_{ij}^k)\, \Theta_{i,j}^k \quad (30)$$

with variable relaxation coefficient σ_{ij}^k. The calculations were complicated by a pronounced tendency to oscillations at the boundaries, $\zeta \to \zeta_W$ and $\eta \to \eta_N$, caused by the rapid growth of $w_{i,j}$ with increasing values of ζ and η. This difficulty was overcome by introducing the following field of relaxation coefficients:

$$\sigma_{ij}^k = \sigma_0^k / [1 + \omega \zeta_j \eta_i (\zeta_j^2 + \eta_i^2)] \quad (31)$$

with values of σ_0^k ranging from 0·5 (at the start) to 1·8 (toward the end of iterations) and with values of ω from 1 (at the start) to 0·1 (toward the end). The computations were accelerated by using low values of i_N and j_W at the beginning and by gradual halving of the grid step after finishing the iteration with a given step α. The input estimates according to Eqs. (*29a,b*) are therefore used only for computations with the largest step.

The computations were carried out on a COMMODORE PC-10 computer with an arithmetic coprocessor 8087 permitting the representation of numbers to 16 decadic digits, and the program was elaborated in TURBO-Pascal. With a maximum number of 6 400 inner grid points, one iteration loop lasted for 40 s. The number of iteration loops, k_{iter}, necessary for stabilization of the grid point values $\Theta_{i,j}$ with a precision to six digits, as well as other data are given in Table I.

RESULTS AND DISCUSSION

By a series of numerical experiments (Table I) it was found that the step $\alpha = 0{\cdot}05$ is sufficient for the determination of the grid point values of the concentration field with a precision to four digits, and the choice of $\zeta_W = 1{\cdot}6$ and $\eta_N = 1{\cdot}6$ permits to determine the parameter $\Psi_S = 0{\cdot}415 \pm 0{\cdot}005$. The basic characteristics of the concentration field (Figs 3–5) show the influence of lateral diffusion on the concentration field close to the lateral edges and justify the choice of $\zeta_W = 1{\cdot}6$ and $\eta_N = 1{\cdot}6$ for the boundaries of the lateral transition region.

To judge the influence of lateral diffusion on the limiting diffusion current, we recall the relations among the three length parameters introduced in the formulation of the problem, λ, L, and $\delta_L = \delta(L)$. According to Eq. (9) we have

$$\frac{\delta_L}{\lambda} = \Gamma(4/3)\,(L/\lambda)^{1/3} = 3^{-1/3}\,\Gamma(4/3)\,\mathrm{Pe}_L^{1/6} \qquad (32)$$

or

$$\frac{L}{\delta_L} = \frac{1}{\Gamma(4/3)}\left(\frac{L}{\lambda}\right)^{2/3} = \frac{1}{3^{2/3}\,\Gamma(4/3)}\,\mathrm{Pe}_L^{1/3}\,, \qquad (33)$$

where

$$\mathrm{Pe}_L = L^2\gamma/D\,. \qquad (34)$$

Hence, for the asymptotic case considered, $\mathrm{Pe}_L \gg 1$, we also have the asymptotic inequalities $L \gg \delta_L \gg \lambda$.

The influence of longitudinal diffusion is for $\mathrm{Pe}_L \gg 1$, according to the known results[6-11], limited to the boundary region at the front and rear edges of a depth not exceeding λ. Our analysis shows that the width of the transition region with

TABLE I
Quantitative characteristics of the calculation

ζ_W	η_N	α	k_{iter}[a]	Ψ_S
1·5	1·5	0·125	90	(0·4)
		0·0625	+116	0·419
		0·03125	+34	0·418
2·0	2·0	0·2	88	(0·5)
		0·1	+61	(0·42)
		0·05	+55	0·415
		0·025	+35	0·413
2·6	2·6	0·26	145	(0·8)
		0·13	+149	(0·5)
		0·065	+137	0·422
		0·0325	+31	0·413

[a] After halving the step, the procedure continues with the obtained estimate of the field $\boldsymbol{\Theta}_{i,j}$.

a pronounced influence of lateral diffusion is of the order of $O(\delta)$. We shall show that for sensors of finite width the influence of lateral diffusion dominates over that of the longitudinal diffusion, which is usually negligible.

In accord with the usual treatment of edge effects[11,13], it is preferable to refer the influence of lateral diffusion on the current increment, ΔI_S, to unit length of the

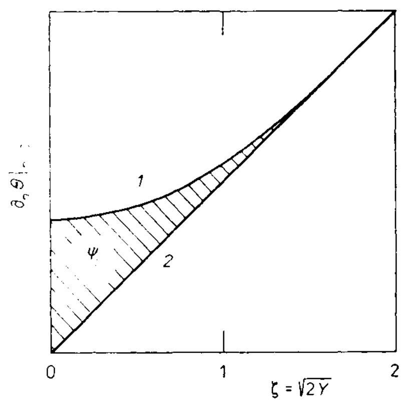


FIG. 3

Longitudinal profile of the surface ($z = 0$) concentration gradient in the region $\mathscr{W}_S$ ($\eta = 0$). *1* From the numerical solution; *2* input estimate according to the Lévêque approximation, Eq. (*29a*). On x-axis for $\sqrt{2Y}$ read $\sqrt{(-2Y)}$

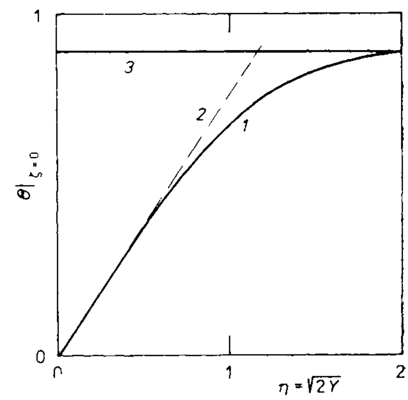
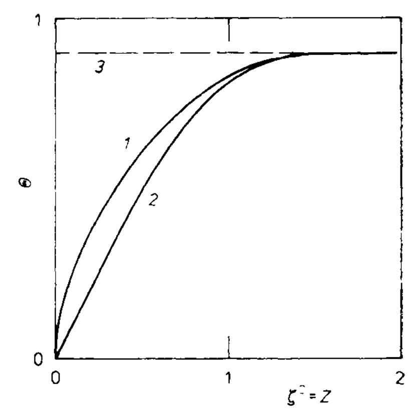


FIG. 4

Longitudinal surface concentration profile ($z = 0$) in the region $\mathscr{N}_S(\zeta = 0)$. *1* From the numerical solution: *2* empirical input estimate according to Eq. (*29b*); *3* limiting value for conditions in the bulk of the liquid

FIG. 5

Transversal concentration profile over the lateral edge ($y = 0, \zeta = \eta$). *1* From the numerical solution; *2* input estimate according to the Lévêque approximation, Eq. (*29a*); *3* limiting value for conditions in the bulk of the liquid



lateral edge:

$$\frac{\Delta I_S}{\Delta x} = FD \int_{-\infty}^{0} \left(\partial_z c\big|_{z=0} - \partial_z c_L\big|_{z=0}\right) \mathrm{d}y = \frac{FDc_0}{\Gamma(4/3)} \Psi_S \,. \tag{35}$$

This increment is independent of x, and for a rectangular sensor of length L we therefore have

$$\Delta I_S = \frac{2FDc_0}{\Gamma(4/3)} L\Psi_S \,. \tag{36}$$

It is known[4,7] that the longitudinal diffusion for $\mathrm{Pe}_L \gg 1$ participates on an appreciable current increment only in the region of the rear edge. Its influence can be expressed by the following estimate of the current increment[11]:

$$\Delta I_T = \frac{FDc_0 h}{\Gamma(4/3)} \left(\frac{\lambda}{L}\right)^{1/3} \Psi_T \,, \tag{37}$$

where $\Psi_T \approx 0{\cdot}23$.

A comparison of the estimates of ΔI_S and ΔI_T leads to the conclusion that for $h = L$, $\mathrm{Pe}_L \gg 1$ the influence of longitudinal diffusion is negligible:

$$\Delta I_T/\Delta I_S \approx 0{\cdot}4 \frac{h}{L} \mathrm{Pe}_L^{-1/6} \,. \tag{38}$$

In the diffusion layer approximation, the total current I_L for the rectangular electrode is given as

$$I_L = \frac{3FDc_0 h}{2\,\Gamma(4/3)} \left(\frac{L}{\lambda}\right)^{2/3} . \tag{39}$$

The correction for lateral diffusion can be expressed according to Eqs (*36*) and (*39*) by the equation

$$1 + \Delta I_S/I_L = 1 + 1{\cdot}2 \frac{L}{h} \mathrm{Pe}_L^{-1/3} \,. \tag{40}$$

For rectangular electrodes[1,14] ($L = 0{\cdot}5$ mm, $h = 0{\cdot}1$ mm) in aqueous solutions ($D \approx 10^{-9}$ m^2 s^{-1}) and at relatively high shear rates ($\gamma \approx 40$ s^{-1}), values of $\mathrm{Pe}_L \approx$ $\approx 10^4$ are sufficiently high to permit neglection of the influence of longitudinal diffusion. Owing to lateral diffusion, however, the current according to Eq. (*40*) increases by about 30%. This effect decreases the sensitivity of the current signal to changing orientation of the rectangular electrode during changing direction of streaming. To measure the direction of streaming at not too high velocities, segmented circular electrodes[1,5,15] are therefore more suitable; nowadays their manufacture has been satisfactorily mastered[15].

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Translated by K. Micka.